Classification of lattice triangles by their first two widths

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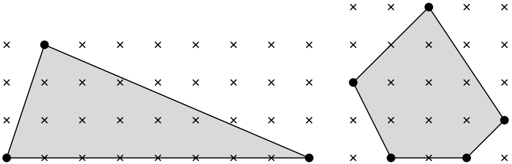
Lattice Polygons

A *lattice polygon* is the convex hull of finitely many points of $\mathbb{Z}^2$.



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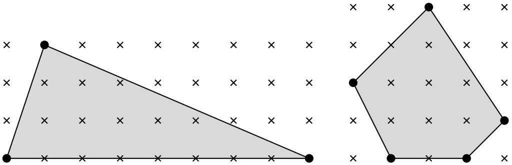

We will classify polygons up to *affine equivalence*, that is, up to change of basis of $\mathbb{Z}^2$ and integral translation. For example

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- reflections
- translations

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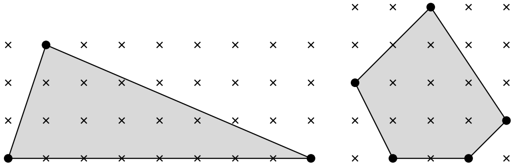

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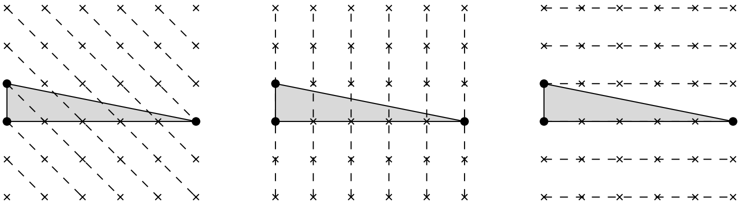
are all affine maps. Such maps preserve the area and lattice point structure of the polygons (how many lattice points in their interior/boundary etc) and are relevant to some applications.

Width

For a lattice polygon P and any normal vector $u \in (\mathbb{Z}^2)^* \cong \mathbb{Z}^2$ we say that the *width of P with respect to u* is

$$\mathrm{width}_u(P) := \max_{x \in P}\{u \cdot x\} - \min_{x \in P}\{u \cdot x\}.$$

The smallest width over all non-zero normal vectors u is called the *width* of P denoted $\mathrm{width}(P)$.



(a) $\mathrm{width}_{(1,1)}(P) = 5$ (b) $\mathrm{width}_{(1,0)}(P) = 5$ (c) $\mathrm{width}_{(0,1)}(P) = 1$

Figure: Widths of $P = \mathrm{conv}((0,0),(5,0),(0,1))$ with respect to different normal vectors.

Motivation

- Iglesias-Valiño and Santos (2021) completed the classification of empty 4-dimensional lattice simplices [IVnS21].
- Morrison and Tewari (2021) classified polygons which contain a lattice point from which all other lattice points are visible [MT21].
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  Apart from one exception all 3-dimensional lattice zonotopes of degree 2 have width 1.

Multi-width

For a lattice polytope $P \subset \mathbb{R}^d$ we can choose linearly independent normal vectors u_1 and u_2 such that the tuple

$$(\text{width}_{u_1}(P), \text{width}_{u_2}(P))$$

is minimal with respect to lexicographic order. This tuple is called the *multi-width* of P denoted $\text{mwidth}(P)$. We call the i-th entry of the multi-width the i-th width of P denoted $\text{width}^i(P)$.

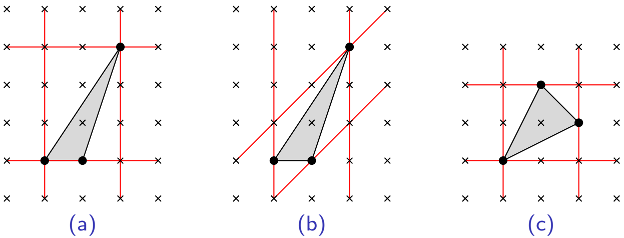

Figure: The multi-width of a polygon gives the dimensions of the smallest parallelogram it is a subset of.

Fact

A lattice polygon P is affine equivalent to a subset of $[0, \text{width}^1(P)] \times [0, \text{width}^2(P)]$. This means that the multi-width records the dimensions of the smallest rectangle P is a subset of up to affine equivalence.

Main Theorem

Let $T(v_1, v_2, v_3)$ denote the triangle with vertices v_1, v_2, v_3. We allow volume zero triangles (where the vertices are colinear) but still record all three vertices.

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Let $0 \leq w_1 \leq w_2$ then we define $\mathcal{T}_{w_1,w_2}$ to be the set of multi-width (w_1, w_2) triangles up to equivalence:

$$\mathcal{T}_{w_1,w_2} := \{T = T(v_1, v_2, v_3) : \text{mwidth}(T) = (w_1, w_2)\}/\sim$$

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Theorem

There is a bijection from $\mathcal{S}_{w_1,w_2}$ to $\mathcal{T}_{w_1,w_2}$ given by the map taking T to its affine equivalence class where we define the set $\mathcal{S}_{w_1,w_2}$ to include the following triangles when $w_1 > 0$

1. $T(0, (w_1, y_1), (0, w_2))$ *where* $y_1 \in [0, w_2 - y_1 \mod w_1]$
2. $T(0, (w_1, y_1), (x_2, w_2))$ *where* $x_2 \in (0, \frac{w_1}{2}]$, $y_1 \in [0, w_1 - x_2]$ *and* $y_1 \geq x_2$ *if* $w_1 = w_2$
3. $T((0, y_0), (w_1, 0), (x_2, w_2))$ *where* $x_2 \in (1, \frac{w_1}{2})$, $y_0 \in (0, x_2)$ *and* $w_1 < w_2$

and if $w_1 = 0$ *then* $\mathcal{S}_{w_1,w_2} := \{T(0, (0, y_1), (0, w_2)) : 0 \leq y_1 \leq \frac{w_2}{2}\}$.

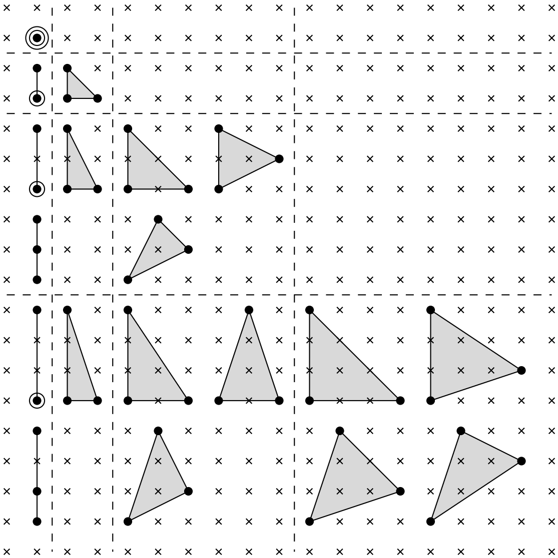


Figure: The triangles $T \in \mathcal{S}_{w_1,w_2}$ where $0 \le w_1 \le w_2 \le 3$. These all satisfy that $\text{width}^1(T) = \text{width}_{(1,0)}(T)$ and $\text{width}^2(T) = \text{width}_{(0,1)}(T)$. When a triangle has multiple identical vertices these are denoted by concentric circles around the first vertex.

Corollary

The cardinality of $\mathcal{T}_{w_1,w_2}$ *is given as follows*

$$|\mathcal{T}_{w_1,w_2}| = \begin{cases} \frac{w_1^2}{2} + 2 & \textit{when } w_1 \textit{ and } w_2 \textit{ even} \\ \frac{w_1^2}{2} + 1 & \textit{when } w_1 \textit{ even and } w_2 \textit{ odd} \\ \frac{w_1^2}{2} + \frac{1}{2} & \textit{when } w_1 \textit{ odd} \end{cases}$$

when $0 < w_1 < w_2$,

$$|\mathcal{T}_{w_1,w_1}| = \begin{cases} \frac{w_1^2}{4} + \frac{w_1}{2} + 1 & \textit{when } w_1 \textit{ even} \\ \frac{w_1^2}{4} + \frac{w_1}{2} + \frac{1}{4} & \textit{when } w_1 \textit{ odd} \end{cases}$$

when $0 < w_1 = w_2$ *and*

$$|\mathcal{T}_{0,w_2}| = \left\lceil \frac{w_2 + 1}{2} \right\rceil$$

when $0 = w_1 \leq w_2$.

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	2	3	3	4	4	5	5	6	6	7
1	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	3	3	4	3	4	3	4	3	4	3	4
3	0	0	0	4	5	5	5	5	5	5	5	5	5
4	0	0	0	0	7	9	10	9	10	9	10	9	10
5	0	0	0	0	0	9	13	13	13	13	13	13	13
6	0	0	0	0	0	0	13	19	20	19	20	19	20

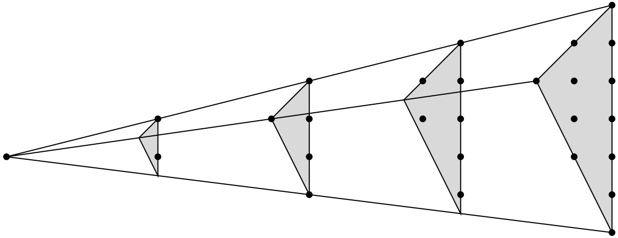
Table: The number of lattice triangles with multi-width (w_1, w_2) up to affine equivalence.

Corollary

The number of lattice triangles which are a subset of $[0,n]^2$ *up to affine equivalence is equal to* $|nQ \cap \mathbb{Z}^4|$ *where*

$$Q := \operatorname{conv}\left(\left(\frac{1}{2},0,0,0\right),\left(0,\frac{1}{2},0,0\right),\left(0,0,\frac{1}{2},0\right),(0,0,0,1),(-1,-1,-1,-1)\right)$$

Furthermore, the generating function of this sequence is the Hilbert series of a degree 8 hypersurface in $\mathbb{P}(1,1,1,2,2,2)$*.*



Sketch proof

Recall: If $w_1 > 0$ then $\mathcal{S}_{w_1,w_2}$ contains the triangles

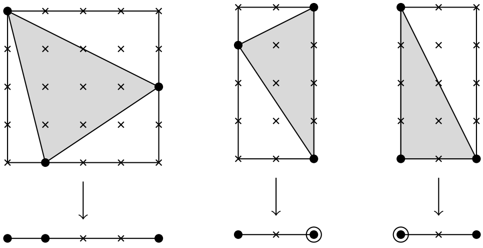
1. $T(0,(w_1,y_1),(0,w_2))$ where $y_1 \in [0, w_2 - y_1 \mod w_1]$
2. $T(0,(w_1,y_1),(x_2,w_2))$ where $x_2 \in (0, \frac{w_1}{2}]$, $y_1 \in [0, w_1 - x_2]$ and $y_1 \geq x_2$ if $w_1 = w_2$
3. $T((0,y_0),(w_1,0),(x_2,w_2))$ where $x_2 \in (1, \frac{w_1}{2})$, $y_0 \in (0, x_2)$ and $w_1 < w_2$

and if $w_1 = 0$ then $\mathcal{S}_{w_1,w_2} := \{T(0,(0,y_1),(0,w_2)) : 0 \leq y_1 \leq \frac{w_2}{2}\}$.

First we can show that all the triangles in $\mathcal{S}_{w_1,w_2}$ have multi-width (w_1, w_2). We know that their widths with respect to $(1,0)$ and $(0,1)$ are w_1 and w_2 respectively so it suffices to show that $\text{width}_u(T) \geq w_2$ for all normal vectors $u = (u_1, u_2)$ linearly independent to $(1,0)$. This can be done by explicitly checking the inequalities in each case.

Surjectivity

Next we must show that any triangle T with multi-width (w_1, w_2) is equivalent to one in $\mathcal{S}_{w_1,w_2}$. We may assume that $T \subseteq [0, w_1] \times [0, w_2]$ and consider the x-coordinates of its vertices. After a reflection we may assume these are 0, x_1 and w_1 for some $x_1 \in [0, \frac{w_1}{2}]$. Therefore we may assume that $T = T((0, y_0), (x_1, y_1), (w_1, y_2))$ for some $y_0, y_1, y_2 \in [0, w_2]$.

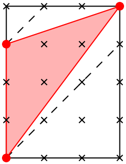


(a) (b) (c)

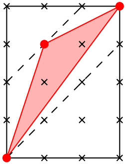
Figure: We first classify the possible x-coordinates of the multi-width (w_1, w_2) triangles then lift to two dimensions in every possible way.

We must have $0, w_2 \in \{y_0, y_1, y_2\}$. By considering the width of T with respect to $(-1, 1)$ we can show that $y_1 = w_2$ and so one of y_0 and y_1 is 0. Considering the width of T w.r.t. $(-1, 1)$ gives the majority of the conditions on the vertices of T that we wanted.

The remaining conditions on triangles in $\mathcal{S}_{w_1, w_2}$ come from removing a few equivalent triangles.



(a) When $x_1 = 0$ we have $y_0 = 0$ and $y_1 = w_2$.



(b) When $0 < y_1 < w_2$ the width is too small w.r.t. $(-1, 1)$.

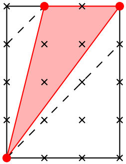


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Figure: Some triangles we eliminate.

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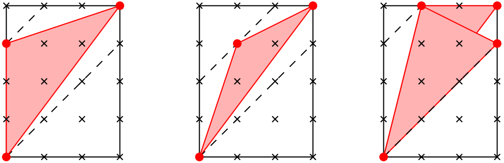

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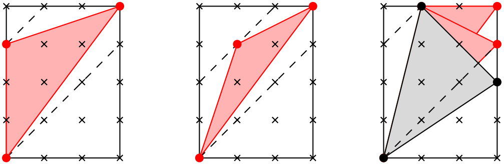

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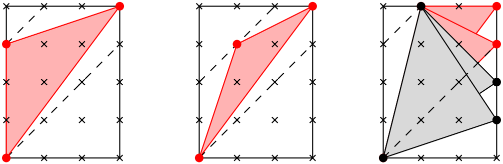

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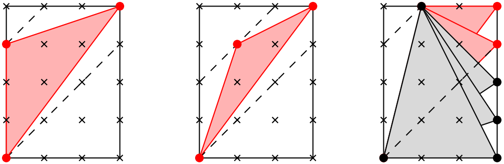

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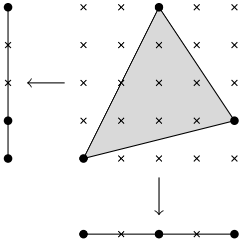
Some significant repeated triangles occur in the case $w_1 = w_2$.
When $w_1 = w_2$ the set of x-coordinates is not uniquely defined. If we require x_1 to be the smallest integer in $[0, w_1]$ such that the vertices of T project down to 0, x_1 and w_1 we get the additional conditions of $\mathcal{S}_{w_1,w_2}$ in the case $w_1 = w_2$. This shows that $T \sim T'$ for some $T' \in \mathcal{S}_{w_1,w_2}$.



(a)

Injectivity

Let $T, T' \in \mathcal{S}_{w_1,w_2}$ such that $T \sim T'$.

When $w_1 < w_2$ it is immediate that the x-coordinates of T and T' are identical. When $w_1 = w_2$ it follows from the fact that the x-coordinates are as small as possible.

The fact that the y-coordinates are equal is then mostly a result of the volumes being equal and so $T = T'$.

The end

Thanks for listening!

References

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Extension to tetrahedra

	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	3	3	3	3	3	3	3	3	3	3
2	0	8	11	12	11	12	11	12	11	12	11	12
3	0	0	13	20	20	20	20	20	20	20	20	20
4	0	0	0	22	35	36	35	36	35	36	35	36
5	0	0	0	0	31	52	52	52	52	52	52	52
6	0	0	0	0	0	44	75	76	75	76	75	76

Table: The number of lattice tetrahedra with multi-width $(1, w_2, w_3)$ up to affine equivalence.

	2	3	4	5	6	7	8	9	10	11	12
2	17	45	47	45	47	45	47	45	47	45	47
3	0	87	178	175	178	175	178	175	178	175	178
4	0	0	161	320	325	320	325	320	325	320	
5	0	0	0	244							

Table: The number of lattice tetrahedra with multi-width $(2, w_2, w_3)$ up to affine equivalence.